

PERSISTENCE OF TRANSLATIONAL SYMMETRY IN THE BCS MODEL WITH RADIAL PAIR INTERACTION

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ABSTRACT. We consider the two-dimensional BCS functional with a radial pair interaction. We show that the translational symmetry is not broken in a certain temperature interval below the critical temperature. In the case of vanishing angular momentum our results carry over to the three-dimensional case.

1. INTRODUCTION

In 1957 Bardeen, Cooper, and Schrieffer published their famous paper with the title "Theory of Superconductivity", which contained the first, generally accepted, microscopic theory of superconductivity. In recognition of this work they were awarded the Nobel prize in 1972. Originally introduced to describe the phase transition from the normal to the superconducting state in metals and alloys, BCS theory can also be applied to describe the phase transition to the superfluid state in cold fermionic gases. In this situation, one has to replace the usual non-local phonon-induced interaction in the gap equation by a local pair potential. Apart from being a paradigmatic model in solid state physics and in the field of cold quantum gases, the BCS theory of superconductivity, that is, the gap equation and the BCS functional show a rich mathematical structure, which has been well recognized. See [21, 2, 22, 23, 20, 24] for works on the gap equation with interaction kernels suitable to describe the physics of conduction electrons in solids and [11, 5, 14, 15, 3, 6, 10] for works that treat the translation-invariant BCS functional with a local pair interaction. One main question in the study of BCS theory is whether the gap equation

$$\Delta(p) = -\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{V}(p-q) \frac{\tanh(E(q)/2T)}{E(q)} \Delta(q) dq, \quad (1.1)$$

with $E(q) = \sqrt{(q^2 - \mu)^2 + |\Delta(q)|^2}$ has a non-trivial solution, that is, one with $\Delta \neq 0$. In [11] it has been demonstrated that, although the gap equation is highly non-linear, this can be decided with the help of a linear criterion. To be more precise, it was shown that the existence of a non-trivial solution of the gap equation is equivalent to the fact that a certain linear operator has a negative eigenvalue. Based on a characterization of the critical temperature in terms of this linear operator, its behavior has been investigated in the limit of small couplings and in the low-density limit, see [5, 15] and [13], respectively. Recently, there has also been considerable interest in the BCS functional with external fields, and in particular, in its connection to the Ginzburg-Landau theory of superconductivity, see [16, 4, 17, 7, 9, 8, 19].

In this work we consider the two-dimensional BCS functional with a radial pair interaction and show that there exists a certain temperature interval below the critical temperature, in which the translational symmetry of the system persists. Our analysis carries over to the three-dimensional case if the Cooper-pairs are in an s-wave state. Prior to this work, such a result was known only in the case of $\hat{V} \leq 0$ and not identically zero, see [18].

2. MAIN RESULTS

We consider a sample of fermionic atoms in a cold gas within the framework of BCS theory. It is convenient to think of the sample as infinite and periodic, since this setting avoids having to deal with boundary conditions at the boundary of the sample. Let us begin by introducing the periodic and the translation-invariant BCS functional. Accordingly, we calculate all energies per unit volume. Thus we choose the lattice \mathbb{Z}^d with the unit cell $[0, 1]^d = \Omega$ and define for a periodic operator A , the trace per unit volume Tr_Ω by $\text{Tr}_\Omega [A] = \text{Tr} [\chi_\Omega A \chi_\Omega]$, where χ_Ω denotes the characteristic function of Ω . Note that we choose \mathbb{Z}^d for mathematical convenience, in fact the lattice can be chosen arbitrarily. The BCS functional at temperature $T \geq 0$, with chemical potential $\mu \in \mathbb{R}$, interaction potential $V \in L^2(\mathbb{R}^d)$ and entropy

$$S(\Gamma) = -\frac{1}{2} \text{Tr}_\Omega [\Gamma \log \Gamma + (1 - \Gamma) \log (1 - \Gamma)],$$

is then given by

$$\mathcal{F}(\Gamma) = \text{Tr}_\Omega [(-\nabla^2 - \mu) \gamma] + \int_{\Omega \times \mathbb{R}^d} V(x - y) |\alpha(x, y)|^2 dx, y - TS(\Gamma). \quad (2.1)$$

Note that the same functional has been considered in [7], where the periodicity was introduced for ease of comparison with the translation-invariant functional. We describe periodic BCS states by self-adjoint operators Γ on $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ which satisfy $0 \leq \Gamma \leq 1$ and can be represented by 2×2 operator-valued matrices of the form

$$\Gamma = \begin{pmatrix} \gamma & \alpha \\ \bar{\alpha} & 1 - \bar{\gamma} \end{pmatrix}, \quad (2.2)$$

where γ and α are periodic operators with period one. In terms of kernels, the latter means that $\gamma(x + u, y + u) = \gamma(x, y)$ and $\alpha(x + u, y + u) = \alpha(x, y)$ for all $u \in \mathbb{Z}^d$ and all $x, y \in \mathbb{R}^d$. In (2.2), $\bar{\alpha} = C\alpha C$, where C denotes complex conjugation. Note that, in particular, $\alpha(x, y) = \alpha(y, x)$ for all $x, y \in \mathbb{R}^d$, due to the self-adjointness of Γ . We call Γ of the form (2.2) an *admissible* BCS state if $\text{Tr}_\Omega (-\nabla^2 + 1)\gamma < \infty$ and denote the set of admissible BCS states by \mathcal{D} . We will, by a slight abuse of notation, write $(\gamma, \alpha) \in \mathcal{D}$, meaning that the BCS state Γ given by (2.2) is admissible.

The translation-invariant BCS functional \mathcal{F}^{ti} is obtained from \mathcal{F} by restricting the set of admissible states to the translation-invariant ones. That is, the kernels of γ and α take the form $\gamma(x, y) = \gamma(x - y)$ and $\alpha(x, y) = \alpha(x - y)$, respectively. We describe translation-invariant BCS states via their momentum representations by 2×2 matrices of the form

$$\Gamma(p) = \begin{pmatrix} \gamma(p) & \hat{\alpha}(p) \\ \hat{\alpha}(\bar{p}) & 1 - \gamma(-p) \end{pmatrix}, \quad (2.3)$$

for $p \in \mathbb{R}^d$, where the bar denotes complex conjugation and $\Gamma(p)$ satisfies $0 \leq \Gamma(p) \leq 1$ for all $p \in \mathbb{R}^d$. The latter translates to $|\hat{\alpha}(p)|^2 \leq \gamma(p)(1 - \gamma(p))$ for almost all $p \in \mathbb{R}^d$ in terms of γ and α . Note that the fact that Γ is self-adjoint implies that α is an even function and that γ is real-valued. We call a translation-invariant BCS state Γ admissible if $\gamma \in L^1(\mathbb{R}^d, (1 + p^2) dp)$ and $\alpha \in H^1(\mathbb{R}^d, dx)$ and define \mathcal{D}^{ti} to be the set of admissible translation-invariant BCS states. For $T \geq 0$ the translation-invariant BCS functional with chemical potential $\mu \in \mathbb{R}$, interaction potential $V \in L^2(\mathbb{R}^d)$ and

entropy S , which we can now write as

$$S(\Gamma) = -\frac{1}{2} \int_{\mathbb{R}^2} \operatorname{tr}_{\mathbb{C}^2} [\Gamma(p) \log \Gamma(p) + (1 - \Gamma(p)) \log (1 - \Gamma(p))] dp,$$

takes the form

$$\mathcal{F}^{\text{ti}}(\Gamma) = \int_{\mathbb{R}^d} (p^2 - \mu) \gamma(p) dp + \int_{\mathbb{R}^2} V(x) |\alpha(x)|^2 dx - TS(\Gamma). \quad (2.4)$$

It was shown in [11, Theorem 1] that a particular linear operator has at least one negative eigenvalue if and only if the normal state is unstable, that is, the energy can be lowered by the formation of Cooper pairs. These results carry over to our setting, where we choose α to be symmetric. In this regard, let us introduce the function $K_T : \mathbb{R}^d \rightarrow \mathbb{R}$ given by

$$K_T(p) = \frac{p^2 - \mu}{\tanh((p^2 - \mu)/(2T))}.$$

Then, $K_T(-i\nabla)$ defines an operator on $L^2(\mathbb{R}^d)$ acting by multiplication by $K_T(p)$ in Fourier space. Observe that K_T is monotone increasing in T , which allows us to define the critical temperature for the BCS functional by

$$T_c := \inf\{T \geq 0 \mid K_T + V \geq 0\}. \quad (2.5)$$

In other words, T_c is the value of T such that the operator $K_T + V$ has zero as lowest eigenvalue.

In this paper, we treat the question whether there is translational symmetry breaking in the BCS model with radial pair interaction V . More precisely, we study the minimization problem

$$\inf \{ \mathcal{F}(\Gamma) \mid \Gamma \in \mathcal{D} \}$$

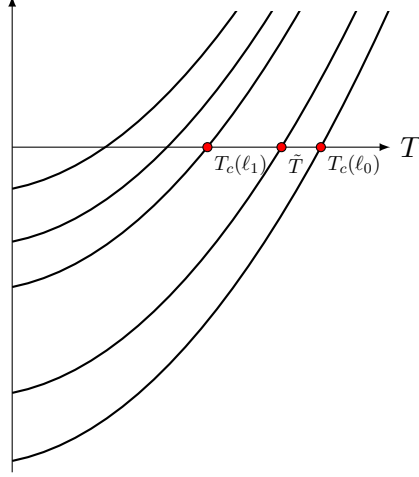
and we are, in particular, concerned with the question whether the infimum of \mathcal{F} is attained by the minimizers of the translation-invariant BCS functional. In three dimensions, this is already known to be the case if $\hat{V} \leq 0$, see [7, 18]. In order to study this question, we consider the BCS functional $\mathcal{F}_\ell^{\text{ti}}$ on the sector of Cooper pair wave functions of angular momentum $\ell \in 2\mathbb{N}$, that we will define in the next paragraph. Our strategy consists of showing that there exists ℓ_0 such that the minimizers of $\mathcal{F}_{\ell_0}^{\text{ti}}$ and \mathcal{F} coincide under certain assumptions. Since $\mathcal{D}^{\text{ti}} \subset \mathcal{D}$ and $\mathcal{F}_\ell^{\text{ti}}$ can be understood as a restriction of \mathcal{F}^{ti} to a smaller domain, this directly implies that the minimizers of \mathcal{F} are translation-invariant.

Let us now introduce the functionals $\mathcal{F}_\ell^{\text{ti}}$. In two dimensions and for radial potential $V \in L^2(\mathbb{R}^2)$ the linear operator $K_T + V$ is rotation invariant and consequently all its eigenstates are of the form

$$\hat{\alpha}_\ell(p) = e^{i\ell\varphi} \sigma_\ell(p), \quad (2.6)$$

for some even $\ell \in \mathbb{Z}$, where $|p|$ and φ denote the polar coordinates of $p \in \mathbb{R}^2$ and σ_ℓ is a radial function. Recall that α is an even function, which requires ℓ to be even. As we will see, the Euler-Lagrange equation of \mathcal{F}^{ti} implies that if $(\gamma, \hat{\alpha}_\ell)$ is a minimizer of \mathcal{F}^{ti} , then γ has to be a radial function. Therefore, we define the BCS functional on the sector of Cooper pair wave functions of angular momentum ℓ as follows. We make an angular decomposition for $(p, q) \mapsto \hat{V}(p - q)$, that is

$$\hat{V}(p - q) = \sum_{\ell \in \mathbb{Z}} \hat{V}_\ell(p, q) e^{i\ell\varphi},$$

FIGURE 1. lowest eigenvalues of $K_T + V$

where φ denotes the angle between p and q . In other words, this means that

$$\hat{V}_\ell(p, q) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\ell\varphi} \hat{V}(p - q) d\varphi. \quad (2.7)$$

Since \hat{V} is a radial function, it only depends on the absolute value of its argument, that is $|p - q| = \sqrt{p^2 + q^2 - 2|p||q|\cos(\varphi)}$, and we conclude that \hat{V}_ℓ is radial in both arguments. Furthermore, observe that $\hat{V}_\ell = \hat{V}_{-\ell}$.

Then, the BCS functional $\mathcal{F}_\ell^{\text{ti}}$ on the sector of Cooper pair wave functions of even angular momentum $\ell \in 2\mathbb{N}$ is given by

$$\mathcal{F}_\ell^{\text{ti}}(\Gamma) = \int_{\mathbb{R}^d} (p^2 - \mu) \gamma(p) dp + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\sigma_\ell(p)} \sigma_\ell(q) \hat{V}_\ell(p, q) dp dq - TS(\Gamma),$$

where V_ℓ is given in (2.7) and Γ is determined by the pair (γ, σ) with radial functions γ and σ . To be more precise, the domain of $\mathcal{F}_\ell^{\text{ti}}$ is given by

$$\mathcal{D}_\ell := \{(\gamma, \sigma_\ell) \mid \gamma, \sigma_\ell \text{ radial and } (\gamma, \alpha_\ell) \in \mathcal{D}^{\text{ti}}, \text{ where } \hat{\alpha}_\ell(p) = e^{i\ell\varphi} \sigma_\ell(p) \text{ for } p \in \mathbb{R}^2\}.$$

Equivalently, $\mathcal{F}_\ell^{\text{ti}}$ can be understood as the restriction of \mathcal{F}^{ti} to pairs $(\gamma, \alpha) \in \mathcal{D}^{\text{ti}}$ with the property that γ is radial and that α is of the form given in (2.6). In Section 3 we will show that $\mathcal{F}_\ell^{\text{ti}}$ has a minimizer.

Next, we characterize the critical temperature $T_c(\ell)$ corresponding to the functionals $\mathcal{F}_\ell^{\text{ti}}$. For this purpose, let us introduce $\mathcal{H} = \{f \in H^1(\mathbb{R}^2, dp) \mid f \text{ radial}\}$. Then the critical temperature $T_c(\ell)$ of $\mathcal{F}_\ell^{\text{ti}}$ is given by

$$T_c(\ell) := \inf \{T \geq 0 \mid (K_T + V_\ell)|_{\mathcal{H}} \geq 0\}. \quad (2.8)$$

One easily sees that T_c is characterized by

$$T_c = \max_{\ell \in 2\mathbb{N}} T_c(\ell),$$

since $T_c \geq T_c(\ell)$ for all ℓ .

Let us now assume that $T_c = T_c(\ell_0)$ and that the lowest eigenvalue of $K_{T_c} + V$ is at most twice degenerate. In other words, we assume the lowest eigenvalue of $K_{T_c} + V$ to be exactly twice degenerate in the case that $\ell_0 \neq 0$ and we assume it to be non-degenerate in the case that $\ell_0 = 0$. An exemplary situation satisfying this assumption is illustrated in Figure 1. The following theorem shows that the translational symmetry in the BCS model persists for $T \in (\tilde{T}, T_c)$ for some $\tilde{T} \in [T_c(\ell_1), T_c)$.

In particular, if $\ell_0 = 0$, the periodic (and the translation-invariant) BCS functional has a, up to a phase, unique radial minimizer (γ_0, α_0) for $T \in (\tilde{T}, T_c)$. If $\ell_0 \neq 0$, the periodic (and the translation-invariant) BCS functional has two minimizers, namely $(\gamma_{\ell_0}, \alpha_{\ell_0})$ and $(\gamma_{\ell_0}, \alpha_{-\ell_0})$, with γ_{ℓ_0} radial and $\alpha_{\pm\ell_0}$ of the form $\hat{\alpha}_{\pm\ell_0}(p) = e^{\pm i\ell_0\varphi} \sigma_{\ell_0}(p)$.

Theorem 1. *Let $V \in L^2(\mathbb{R}^2)$ with $\hat{V} \in L^r(\mathbb{R}^2)$, where $r \in [1, 2)$, be radial and such that $T_c > 0$. Suppose that $T_c = T_c(\ell_0)$ and that the lowest eigenvalue of $K_{T_c} + V$ is at most twice degenerate. If*

$$(\gamma_{\ell_0}, \sigma_{\ell_0}) \in \mathcal{D}_{\ell_0}$$

minimizes $\mathcal{F}_{\ell_0}^{\text{ti}}$, then there exists $\tilde{T} < T_c$ such that

$$(\gamma_{\ell_0}, \alpha_{\ell_0}) \text{ and } (\gamma_{\ell_0}, \alpha_{-\ell_0}) \in \mathcal{D}^{\text{ti}},$$

where $\hat{\alpha}_{\pm\ell_0}(p) = e^{\pm i\ell_0\varphi} \sigma_{\ell_0}(p)$, minimize the BCS functional \mathcal{F} for $T \in [\tilde{T}, T_c)$. For $T \in (\tilde{T}, T_c)$ these are the only minimizers of \mathcal{F} up to a phases in front of α_{ℓ_0} and $\alpha_{-\ell_0}$.

Remark 2.1. We want to emphasize that \tilde{T} is determined by the lowest nonzero eigenvalue of $K_{T_c} + V$. More precisely, \tilde{T} is given as the value of T such that the second eigenvalue (counted without multiplicities) of $K_T + V$ is zero, which is illustrated in Figure 1. In particular, if in addition to the assumption above, the second eigenvalue of $K_{T_c} + V$ lies in the sector of angular momentum $\ell_1 \neq \ell_0$, one can show that $\tilde{T} = T_c(\ell_1)$.

Remark 2.2. Note that the Fourier transform of the function α , where $\alpha(x) = e^{i\ell\varphi} \tilde{\sigma}_{\ell}(|x|)$ is given by

$$\hat{\alpha}(p) = e^{i\ell\theta} e^{i\ell\pi/2} \int_0^\infty \tilde{\sigma}_{\ell}(|x|) J_{\ell}(|p||x|) |x| dx,$$

where θ corresponds to p and J_{ℓ} denotes the ℓ -th Bessel function.

Remark 2.3. An important step in the proof of Theorem 1 is comparing the minimizers of the BCS functional $\mathcal{F}_{\ell_0}^{\text{ti}}$ on the sector of Cooper pair wave functions of angular momentum ℓ_0 with the minimizers of the periodic BCS functional \mathcal{F} . The crucial tool for that comparison will be the relative entropy inequality, [7, Lemma 5].

Remark 2.4. It is shown in [10], amongst other things, that for every $\ell \in 2\mathbb{N}$ one can find a radial potential such that the ground state of $K_{T_c} + V$ is of angular momentum ℓ . In the case of weak coupling, that is for $K_T + \lambda V$, where $\lambda \in \mathbb{R}$ is small enough, the methods of [5, 15] can be applied to determine the angular momentum ℓ_0 of the ground state of $K_{T_c} + V$. An application of these methods reduces the problem of finding the eigenvalues of $K_T + \lambda V$, for λ small enough, to finding the eigenvalues of a simple matrix, that only depends on the behavior of \hat{V} on the Fermi sphere. This is easily solvable numerically. In particular, one sees, that the eigenvalues are in one-to-one correspondence to the eigenvalues of the matrix $(\langle \psi_n, \hat{V} \psi_m \rangle)_{n,m \geq 0}$, where $\psi_n(p) = e^{in\varphi}$. Moreover, if the lowest eigenvalue of this matrix is at most twice degenerate one is in the situation described in Remark 2.1, i.e. $\tilde{T} = T_c(\ell_1)$.

Remark 2.5. In the non-interacting case, that is $V = 0$, the minimizer of the BCS function \mathcal{F} is given by

$$\Gamma_n = \begin{pmatrix} \gamma_n & 0 \\ 0 & 1 - \overline{\gamma_n} \end{pmatrix},$$

where $\gamma_n = (1 + \exp((-\nabla^2 - \mu)/T))^{-1}$. We refer to Γ_n as the normal state of the system. Let us assume that we are in the situation of Remark 2.1. Having in mind that the linear operator $K_T + V$, which characterizes T_c , is related to the second variation of \mathcal{F} at the normal state Γ_n in the direction of α by

$$\left. \frac{d^2}{dt^2} \mathcal{F}(\gamma_n, t\alpha) \right|_{t=0} = 2\langle \alpha, (K_T + V)\alpha \rangle,$$

one can understand Theorem 1 as follows. We find $T < T_c$ such that $K_T + V$ has exactly one negative eigenvalue λ_0 . Hence the second variation is smallest (and, in particular, negative) if α is an element of the eigenspace of λ_0 and one could therefore hope to find a minimizer of \mathcal{F} which lies approximately in this eigenspace. In fact, Theorem 1 states that the minimizers of \mathcal{F} for temperatures T in a certain interval below T_c lie in exactly one specific sector of angular momentum $\pm\ell_0$. For $T = T_c(\ell_1)$ the next eigenvalue λ_1 and its eigenspace become important, since now also elements of the eigenspace of λ_1 are candidates to lower the energy.

In the special case $\ell_0 = 0$, Theorem 1 also holds in three dimensions. If the eigenfunction of $K_{T_c} + V$ to the eigenvalue zero lies in any other than the radial sector, one has to deal with the spherical harmonics, which carry with them a different mathematical structure.

The following theorem generalizes an observation made in [7], that has been also explained in [18], namely that for potentials with non-positive Fourier transform, that is $\hat{V} \leq 0$, the translational symmetry in the BCS model is not broken.

Theorem 2. *Let $V \in L^2(\mathbb{R}^3)$ with $\hat{V} \in L^r(\mathbb{R}^3)$ for some $r \in [1, 12/7)$ be radial and such that $T_c > 0$. Assume that zero is a non-degenerate eigenvalue of $K_{T_c} + V$, that is the corresponding eigenfunction is radial. Then, there exists $\tilde{T} < T_c$ such that the minimizer of the BCS functional \mathcal{F} for $T \in [\tilde{T}, T_c)$ is given by a pair (γ_0, α_0) , where γ_0 and α_0 are radial functions. Moreover, (γ_0, α_0) is, up to phases, the only minimizer of \mathcal{F} for $T \in (\tilde{T}, T_c)$.*

Remark 2.6. Note that $\hat{V} \leq 0$ implies that the ground state of $K_{T_c} + V$ is radial in all dimensions. Hence, the assumption that $K_{T_c} + V$ has a non-degenerate lowest eigenvalue is always satisfied for interaction potentials V with this property.

3. PREPARATIONS

The proof of Theorem 2 works similarly to the proof of Theorem 1. In order to prove Theorem 1 we will show that there exists $\ell_0 \in 2\mathbb{N}$, such that the minimizers of $\mathcal{F}_{\ell_0}^{\text{ti}}$ also minimize \mathcal{F} . The following lemma lays the basis for this approach.

Lemma 3.1. *The BCS functional $\mathcal{F}_{\ell}^{\text{ti}}$ is bounded from below and attains its minimum.*

Proof. Boundedness from below of $\mathcal{F}_{\ell}^{\text{ti}}$ follows from the fact that \mathcal{F}^{ti} is bounded from below. As in the proof of [11, Lemma 1] we find a minimizing sequence $(\gamma_{\ell}^{(n)}, \sigma_{\ell}^{(n)})$ in \mathcal{D}_{ℓ} that converges strongly in $L^p(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ to (γ, σ) for some $p \in (1, \infty)$, as n tends to infinity. It is an easy consequence that $(\gamma, \sigma) \in \mathcal{D}_{\ell}$. \square

The Euler-Lagrange equation of $\mathcal{F}_{\ell}^{\text{ti}}$ takes the same form as the Euler-Lagrange equation of \mathcal{F}^{ti} , which will play an important role in the proof. The derivation of the Euler-Lagrange equation of \mathcal{F}^{ti} given in [18, Proposition 3.1] translates to the case of $\mathcal{F}_{\ell}^{\text{ti}}$. Therefore, we will not rewrite the proof here, but only give the

Euler-Lagrange equation of $\mathcal{F}_\ell^{\text{ti}}$ and its various forms. Let us emphasize here, that the Euler-Lagrange equations of \mathcal{F}^{ti} and of $\mathcal{F}_\ell^{\text{ti}}$ take the same form. The Euler-Lagrange equation of $\mathcal{F}_\ell^{\text{ti}}$, $\ell \in 2\mathbb{N}$, can be conveniently expressed as an equation for the function $\Delta_\ell : \mathbb{R}^2 \rightarrow \mathbb{R}$. We abbreviate $E_\ell(p) = \sqrt{(p^2 - \mu)^2 + |\Delta_\ell(p)|^2}$ and write the Euler-Lagrange equation of $\mathcal{F}_\ell^{\text{ti}}$ as

$$\Delta_\ell(p) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{V}_\ell(p, q) \frac{\Delta_\ell(q)}{E_\ell(q)} \tanh\left(\frac{E_\ell(q)}{2T}\right) dq. \quad (3.1)$$

Then,

$$\sigma_\ell(p) = -\frac{\Delta_\ell(p)}{2E_\ell(p)} \tanh\left(\frac{E_\ell(p)}{2T}\right). \quad (3.2)$$

Note that the last equation implies that Δ_ℓ is a radial function. In the following we rewrite the Euler-Lagrange equation (3.1) in terms of Γ_ℓ . Abbreviating $k(p) = p^2 - \mu$, we set

$$H_{\Delta_\ell}(p) = \begin{pmatrix} \frac{k(p)}{\Delta_\ell(p)} & \Delta_\ell(p) \\ \Delta_\ell(p) & -k(p) \end{pmatrix}, \quad (3.3)$$

for $p \in \mathbb{R}^2$. For $T > 0$, the Euler-Lagrange equation of the functional $\mathcal{F}_\ell^{\text{ti}}$, can be written as

$$\Gamma_\ell(p) = \frac{1}{1 + e^{H_{\Delta_\ell}(p)/T}}. \quad (3.4)$$

The existence of a non-trivial solution of the the so-called BCS gap equation of $\mathcal{F}_{\ell_0}^{\text{ti}}$ in the relevant temperature interval is a key ingredient of our proof of Theorem 1. In order to derive the BCS gap equation of $\mathcal{F}_\ell^{\text{ti}}$ from (3.4) let us introduce the function $K_T^{\Delta_\ell}$, which is defined for $T > 0$ by

$$K_T^{\Delta_\ell}(p) = \frac{E_\ell(p)}{\tanh(E_\ell(p)/(2T))}.$$

Then $K_T^{\Delta_\ell}(-i\nabla)$ defines an operator on $L^2(\mathbb{R}^2)$ acting by multiplication with $K_T^{\Delta_\ell}(p)$ in Fourier space. Calculations explicitly given in [18] show that (3.4) is equivalent to

$$\gamma_\ell(p) = \frac{1}{2} - \frac{p^2 - \mu}{2K_T^{\Delta_\ell}(p)}, \quad (3.5)$$

$$\sigma_\ell(p) = -\frac{\Delta_\ell(p)}{2K_T^{\Delta_\ell}(p)}. \quad (3.6)$$

Making use of the relation between Δ_ℓ and σ_ℓ given in (3.1) and (3.2) one easily derives the BCS gap equation of $\mathcal{F}_\ell^{\text{ti}}$,

$$\left(K_T^{\Delta_\ell} + V_\ell\right) \sigma_\ell = 0, \quad (3.7)$$

from (3.6). We will also make use of this equation in the form

$$\left(K_T^{\Delta_\ell} + V\right) \alpha_\ell = 0, \quad (3.8)$$

where α_ℓ is of the form (2.6). Let us emphasize that (3.1), (3.4), (3.5) and (3.6), (3.7) and (3.8) are equivalent. In [11] it was shown that \mathcal{F}^{ti} is bounded from below and attains its infimum on \mathcal{D}^{ti} in three dimensions. The same results hold in two dimensions by analogous arguments, which provides a solution of the BCS gap equation.

4. PROOF OF THEOREM 1 AND THEOREM 2

We begin with the proof of Theorem 1. Let $(\gamma_{\ell_0}, \sigma_{\ell_0}) \in \mathcal{D}_{\ell_0}$ be a minimizer of $\mathcal{F}_{\ell_0}^{\text{ti}}$ and assume $T_c = T_c(\ell_0)$. Let Γ_{ℓ_0} be the BCS state given by the pair $(\gamma_{\ell_0}, \alpha_{\ell_0})$ with $\alpha_{\ell_0}(p) = e^{i\ell_0\varphi} \sigma_{\ell_0}(p)$. Our aim is to show that the inequality $\mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_{\ell_0}) \geq 0$ holds for all $\Gamma \in \mathcal{D}$. We will use a generalization of the trace per unite volume, which for a periodic operator A on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ is defined by

$$\text{Tr}_0[A] = \text{Tr}_\Omega[P_0 A P_0 + Q_0 A Q_0]$$

with

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that if A is locally trace class, then $\text{Tr}_0[A] = \text{Tr}_\Omega[A]$.

We begin by calculating the difference $\mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_\ell)$, where Γ_ℓ corresponds to a minimizer of $\mathcal{F}_\ell^{\text{ti}}$ as described above, and find that

$$\begin{aligned} & \mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_\ell) \\ &= \text{Tr}_\Omega [(-\nabla^2 - \mu)(\gamma - \gamma_\ell)] + \int_{\Omega \times \mathbb{R}^2} V(x-y) (|\alpha(x,y)|^2 - |\alpha_\ell(x,y)|^2) \, d(x,y) \\ & \quad - T(S(\Gamma) - S(\Gamma_\ell)). \end{aligned} \tag{4.1}$$

First, we complete the square in the difference of the interaction terms, which yields

$$\begin{aligned} & \int_{\Omega \times \mathbb{R}^2} V(x-y) (|\alpha(x,y)|^2 - |\alpha_\ell(x,y)|^2) \, d(x,y) \\ &= \int_{\Omega \times \mathbb{R}^2} V(x-y) (|\alpha(x,y) - \alpha_\ell(x,y)|^2) \, d(x,y) \\ & \quad - 2 \int_{\Omega \times \mathbb{R}^2} V(x-y) \left(|\alpha_\ell(x,y)|^2 - \text{Re} \left(\overline{\alpha(x,y)} \alpha_\ell(x,y) \right) \right) \, d(x,y). \end{aligned}$$

Next, we combine the second term on the right hand side and the first term on the right hand side of (4.1). Inserting the relation between Δ_ℓ and α_ℓ , we see that

$$\begin{aligned} & \text{Tr}_\Omega [(-\nabla^2 - \mu)(\gamma - \gamma_\ell)] \\ & \quad + 2 \text{Re} \int_{\Omega \times \mathbb{R}^2} V(x-y) \left(\alpha_\ell(x,y) \overline{\alpha(x,y)} - |\alpha_\ell(x,y)|^2 \right) \, d(x,y) \\ &= \frac{1}{2} \text{Tr}_0 [H_{\Delta_\ell} (\Gamma - \Gamma_\ell)], \end{aligned}$$

where H_{Δ_ℓ} is given as before, see (3.3).

At this point, it turns out to be convenient to introduce the relative entropy \mathcal{H} , which for two BCS states $\Gamma, \tilde{\Gamma} \in \mathcal{D}$ is given by

$$\mathcal{H}(\Gamma, \tilde{\Gamma}) = \text{Tr}_0 \left[\Gamma \left(\log \Gamma - \log \tilde{\Gamma} \right) + (1 - \Gamma) \left(\log(1 - \Gamma) - \log(1 - \tilde{\Gamma}) \right) \right].$$

The fact that $H_{\Delta_\ell}/T = \log(1 - \Gamma_\ell) - \log \Gamma_\ell$ yields the following statement.

Lemma 4.1. *Let $(\gamma_\ell, \sigma_\ell) \in \mathcal{D}_\ell$ be a minimizer of $\mathcal{F}_\ell^{\text{ti}}$ and let Γ_ℓ be given by the pair $(\gamma_\ell, \alpha_\ell)$ where $\alpha_\ell(p) = e^{i\ell\varphi} \sigma_\ell(p)$. Then*

$$\mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_\ell) = \frac{T}{2} \mathcal{H}(\Gamma, \Gamma_\ell) + \int_{\Omega \times \mathbb{R}^2} V(x-y) |\alpha(x,y) - \alpha_\ell(x,y)|^2 \, d(x,y)$$

for all $\Gamma \in \mathcal{D}$, where $\alpha = (\Gamma)_{12}$.

Based on this identity, we estimate $\mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_{\ell_0})$ from below by applying the relative entropy inequality [7, 12].

Proposition 4.2. *Let $(\gamma_\ell, \sigma_\ell) \in \mathcal{D}_\ell$, be a minimizer of $\mathcal{F}_\ell^{\text{ti}}$, let Γ_ℓ be as in Lemma 4.1 and denote $V_y(x) = V(x - y)$. Then, for all $\Gamma \in \mathcal{D}$, with $\alpha = (\Gamma)_{12}$,*

$$\mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_\ell) \geq \int_{\Omega} \left\langle \alpha, \left(K_T^{\Delta_\ell} + V_y(x) \right)_x \alpha \right\rangle_{L^2(\mathbb{R}^2, dx)} dy + \text{Tr}_{\Omega} K_T^{\Delta_\ell} (\Gamma - \Gamma_\ell)^2.$$

Here, we understand $(K_T^{\Delta_\ell} + V_y(x))_x$ as an operator acting on the x -coordinate of $\alpha(x, y)$.

Proof. The claimed estimate is a consequence of an inequality for the relative entropy that has been proven in [7, Lemma 5]. An application of this inequality yields

$$\begin{aligned} \mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_\ell) &\geq \frac{1}{2} \text{Tr}_{\Omega} \left[(\Gamma - \Gamma_\ell) \frac{H_{\Delta_\ell}}{\tanh(H_{\Delta_\ell}/(2T))} (\Gamma - \Gamma_\ell) \right] \\ &\quad + \int_{\Omega \times \mathbb{R}^2} V(x - y) |\alpha(x, y) - \alpha_\ell(x, y)|^2 dx dy. \end{aligned}$$

The fact that $x \mapsto x(\tanh(x/2))^{-1}$ is an even function and

$$H_{\Delta_\ell}^2(p) = \mathbb{I}_{\mathbb{C}^2} E_\ell^2(p)$$

is diagonal, implies the statement. \square

Next, we show that the operator $K_T^{\Delta_{\ell_0}} + V$ is nonnegative for $T \in [\tilde{T}, T_c)$.

Proposition 4.3. *Assume $\hat{V} \in L^r(\mathbb{R}^2)$ for some $r \in [1, 2)$. If the lowest eigenvalue of $K_{T_c} + V$ is at most twice degenerate then there exists $\tilde{T} < T_c$ such that $K_T^{\Delta_{\ell_0}} + V$ is nonnegative as an operator on $L^2(\mathbb{R}^2)$ for all $T \in [\tilde{T}, T_c)$.*

Remark 4.4. The non-negativity of $K_T^{\Delta} + V$ in our situation is the key ingredient to our proof of Theorem 1. As we will show by a rearrangement argument, $K_T^{\Delta} + V$ has a negative eigenvalue if the considered Cooper pair wave functions α do not have the property that $|\alpha|$ is radial. In particular, this means that our strategy cannot be extended to situations where the ground state of $K_{T_c} + V$ is not radial.

To see this, let $R \in SO(3)$ and $U(R)f(p) = f(R^{-1}p)$ for all $f \in L^2(\mathbb{R}^3)$. We claim that there exists $R \in SO(3)$ such that

$$\langle U(R)\alpha, K_T^{\Delta} U(R)\alpha \rangle < \langle \alpha, K_T^{\Delta} \alpha \rangle.$$

Inserting the relation $\alpha(p) = -\Delta(p)/(2K_T^{\Delta}(p))$, which is implied by the Euler-Lagrange equation of \mathcal{F} (see Section 3) we find that

$$\begin{aligned} \langle U(R)\alpha, K_T^{\Delta} U(R)\alpha \rangle &= \frac{1}{4} \int_{\mathbb{R}^3} \frac{|\Delta(p)|^2}{K_T^{\Delta}(p)^2} K_T^{\Delta}(Rp) dp \\ &= \frac{1}{4} \int_0^\infty \int_{\Omega_r} \frac{|\Delta(p)|^2}{K_T^{\Delta}(p)^2} K_T^{\Delta}(Rp) d\omega(p) r^2 dr, \end{aligned}$$

where Ω_r denotes the sphere with radius r and $d\omega$ denotes uniform measure on Ω_r . On Ω_r , that is for fixed radius $|p|$, we can understand $|\Delta(p)|^2/K_T^{\Delta}(p)^2$ as a function f that depends only on $\Delta(p)$. In the same way we can also find a function g such

that $K_T^\Delta(Rp) = g(\Delta(Rp))$ for all $p \in \mathbb{R}^3$. Then f and g both are strictly increasing functions. Thus, denoting by Δ^* the cap rearrangement of Δ , see [1], we find that

$$\begin{aligned} \int_0^\infty \int_{\Omega_r} \frac{|\Delta(p)|^2}{K_T^\Delta(p)^2} K_T^\Delta(Rp) d\omega(p) r^2 dr &= \int_0^\infty \int_{\Omega_r} f(\Delta(p)) g(\Delta(Rp)) d\omega(p) r^2 dr \\ &\leq \int_0^\infty \int_{\Omega_r} f(\Delta^*(p)) g(\Delta^*(p)) d\omega(p) r^2 dr. \end{aligned} \quad (4.2)$$

Note that $\Delta(R\cdot)$ and Δ^* are equimeasurable. The crucial argument is the monotonicity of f and g , which implies

$$\int_0^\infty \int_{\Omega_r} f(\Delta(p)) g(\Delta(p)) d\omega(p) r^2 dr = \int_0^\infty \int_{\Omega_r} f(\Delta^*(p)) g(\Delta^*(p)) d\omega(p) r^2 dr.$$

In particular, this equality implies that we can find $R \in SO(3)$ such that inequality (4.2) is strict and hence proves the claim.

The proof of Proposition 4.3 is based on spectral perturbation theory and relies on the fact that $K_T^{\Delta_{\ell_0}} + V \rightarrow K_{T_c} + V$, while $\Delta_{\ell_0}(T) \rightarrow 0$, in norm resolvent sense for $T \rightarrow T_c$. We will derive this convergence from the following lemmas. In order to simplify the notation we write $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leq cb$. Moreover, we denote by $\|\cdot\|$ the operator norm and by $\|\cdot\|_r$ the $L^r(\mathbb{R}^2)$ -norm.

Lemma 4.5. *Let $T \in (0, T_c)$. The operators $K_{T_c} - K_T$ and $K_T^{\Delta_{\ell_0}} - K_T$ are bounded. More precisely, $\|K_{T_c} - K_T\| \lesssim (T_c - T)$ and $\|K_T^{\Delta_{\ell_0}} - K_T\| \lesssim \|\Delta_{\ell_0}\|_\infty$. Moreover, $K_{T_c} - K_T \geq 0$ and $K_T^{\Delta_{\ell_0}} - K_T \geq 0$.*

Proof. In the proof we abbreviate $A_T := K_{T_c} - K_T$ and $B_T := K_T^{\Delta_{\ell_0}} - K_T$. Notice that

$$K_T^{\Delta_{\ell_0}}(p) = \frac{\sqrt{k(p)^2 + |\Delta_{\ell_0}(p)|^2}}{\tanh\left(\sqrt{k(p)^2 + |\Delta_{\ell_0}(p)|^2}/(2T)\right)}$$

is an increasing function in T for fixed Δ_{ℓ_0} and vice versa. Hence $A_T \geq 0$ and $B_T \geq 0$. Both, A_T and B_T are pseudo-differential operators and by a slight abuse of notation we denote by $A_T(p)$ and $B_T(p)$ the symbols of A_T and B_T , respectively. In the following we abbreviate $T_c - T = \delta T$ and

$$I_T = \frac{1}{T} - \frac{1}{T_c}.$$

A simple calculation yields

$$A_T(p) = \int_0^1 \frac{I_T k(p)^2}{2 \sinh^2(k(p)/(2T_c) + t I_T k(p)/2)} dt.$$

Obviously, for large $|p|$ the smooth function $A : p \mapsto A(p)$ and all its derivatives have exponential decay. Moreover, $|I_T| \lesssim T_c - T$ implies $\|A_T\| \lesssim T_c - T$. In order to derive an analogous representation for $B_T(p)$ we define

$$f(x) := \frac{d}{dx} \frac{x}{\tanh(x/(2T))} = \frac{T \sinh(x/T) - x}{2T \sinh^2(x/(2T))} \quad (4.3)$$

as well as

$$\delta E_{\ell_0}(p) = \sqrt{k(p)^2 + |\Delta_{\ell_0}(p)|^2} - |k(p)|. \quad (4.4)$$

A straightforward calculation shows that

$$B_T(p) = \delta E_{\ell_0}(p) \int_0^1 f(|k(p)| + t\delta E_{\ell_0}(p)) dt. \quad (4.5)$$

Since the function f defined in (4.3) is bounded by 1, we find that $|B_T(p)| \leq |\delta E_{\ell_0}(p)|$ for all $p \in \mathbb{R}^2$. It can be seen directly from the definition of $\delta E_{\ell_0}(p)$, see (4.4), that $|\delta E_{\ell_0}(p)| \leq |\Delta_{\ell_0}(p)|$ for all $p \in \mathbb{R}^2$, which implies $|B_T(p)| \leq |\Delta_{\ell_0}(p)|$ for all $p \in \mathbb{R}^2$. \square

Lemma 4.6. *Let $T \in (0, T_c)$. If α_{ℓ_0} is a solution of the BCS gap equation of $\mathcal{F}_{\ell_0}^{\text{ti}}$, then $\|(1 + p^2)^{1/4} \hat{\alpha}_{\ell_0}\|_4^4 \lesssim \langle \alpha_{\ell_0}, (K_T^{\Delta_{\ell_0}} - K_T) \alpha_{\ell_0} \rangle$.*

Proof. We will make use of the following observation, which is implied by the fact that the function $|\Delta_{\ell_0}| \mapsto |\Delta_{\ell_0}|/K_T^{\Delta_{\ell_0}}$ is strictly increasing. Note that the combination of (3.1) and (3.2) shows that

$$\|\Delta_{\ell_0}\|_{\infty} \leq \|V\|_2 \|\hat{\alpha}_{\ell_0}\|_2. \quad (4.6)$$

We will abbreviate $\|V\|_2 \|\hat{\alpha}_{\ell_0}\|_2$ by $c(\alpha_{\ell_0})$ in the following. Thus, together with (3.6), the just mentioned monotonicity of $|\Delta_{\ell_0}|/K_T^{\Delta_{\ell_0}}$ implies that

$$|\hat{\alpha}_{\ell_0}(p)| \leq \frac{c(\alpha_{\ell_0})}{2K_T^{c(\alpha_{\ell_0})}(p)}$$

for all $p \in \mathbb{R}^2$. By taking the square and integrating, we see that

$$1 \leq \frac{\|V\|_2^2}{4} \int_{\mathbb{R}^2} \left(K_T^{c(\alpha_{\ell_0})}(p) \right)^{-2} dp.$$

Next, we use that $\tanh(x) \leq 1$ for all x , which leads to

$$1 \leq \frac{\|V\|_2^2}{4} \int_{\mathbb{R}^2} ((p^2 - \mu)^2 + \|V\|_2^2 \|\hat{\alpha}_{\ell_0}\|_2^2)^{-1} dp$$

We may assume that $\|V\|_2^2 \|\alpha_{\ell_0}\|_2^2 \geq \mu^2$ and conclude that

$$1 \leq \frac{\|V\|_2^2}{4} \int_{\mathbb{R}^2} (p^4/2 - \mu^2 + \|V\|_2^2 \|\hat{\alpha}_{\ell_0}\|_2^2)^{-1} dp.$$

From this estimate one easily derives that

$$\|\hat{\alpha}_{\ell_0}\|_2^2 \leq \frac{\|V\|_2^2 \pi^4}{32} + \frac{\mu^2}{\|V\|_2^2}.$$

Making use of (4.6), we see that this directly implies that

$$\|\Delta_{\ell_0}\|_{\infty}^2 \leq \frac{\|V\|_2^4 \pi^4}{32} + \mu^2. \quad (4.7)$$

In other words, there exists a constant $m > 0$ that only depends on V and μ , such that $|\Delta_{\ell_0}(p)| < m$ for all $p \in \mathbb{R}^2$. In particular, m does not depend on T .

We have to estimate $K_T^{\Delta_{\ell_0}} - K_T$ from below. We recall that $|\Delta_{\ell_0}| \mapsto K_T^{\Delta_{\ell_0}}/|\Delta_{\ell_0}|^2$ is decreasing. Having in mind that $K_T^{\Delta} - K_T$ behaves like $|\Delta|^2$ for small $|\Delta|$ we thus estimate

$$\frac{K_T^{\Delta_{\ell_0}} - K_T}{|\Delta_{\ell_0}|^2} |\Delta_{\ell_0}|^2 \gtrsim \left(\frac{K_T^m - K_T}{m^2} \right) |\Delta_{\ell_0}|^2.$$

Abbreviating $y_t = \sqrt{k(p)^2 + tm^2}/(2T)$ we find that

$$\begin{aligned} K_T^{\Delta_{\ell_0}}(p) - K_T(p) &= 2T \int_0^1 \frac{d}{dt} \frac{y_t}{\tanh(y_t)} dt \\ &= \frac{m^2}{4T} \int_0^1 \left(\frac{1}{y_t \tanh(y_t)} - \frac{1}{\sinh^2(y_t)} \right) dt. \end{aligned} \quad (4.8)$$

As one easily sees, the function

$$g(y) = \frac{1}{y} \frac{1}{\tanh(y)} - \frac{1}{\sinh^2(y)}$$

is decreasing, which implies

$$K_T^{\Delta_{\ell_0}}(p) - K_T(p) \gtrsim \frac{m^2}{4T} \left(\frac{1}{y_1 \tanh(y_1)} - \frac{1}{\sinh^2(y_1)} \right).$$

Moreover, g is bounded from below by $g(y) \geq 2/3(1+y)^{-1}$. Together with (4.8) this shows that

$$K_T^{\Delta_{\ell_0}}(p) - K_T(p) \gtrsim |\Delta_{\ell_0}(p)|^2 \frac{1}{1+p^2}. \quad (4.9)$$

Next, we make use of the Euler-Lagrange equation of $\mathcal{F}_{\ell_0}^{\text{ti}}$, that is the relation $|\Delta_{\ell_0}(p)| = 2K_T^{\Delta_{\ell_0}}(p)|\hat{\alpha}_{\ell_0}(p)|$. Inserting this identity in (4.9) we see that

$$K_T^{\Delta_{\ell_0}}(p) - K_T(p) \gtrsim \left(K_T^{\Delta_{\ell_0}}(p) \right)^2 \frac{|\hat{\alpha}_{\ell_0}(p)|^2}{1+p^2} \gtrsim (1+p^2) |\hat{\alpha}_{\ell_0}(p)|^2,$$

which implies the statement. \square

Lemma 4.7. *Let $T \in (0, T_c)$. If α_{ℓ_0} is a solution of the BCS gap equation of $\mathcal{F}_{\ell_0}^{\text{ti}}$, then $\|\alpha_{\ell_0}\|_2 \lesssim (T_c - T)^{1/2}$. In particular, $\|\Delta_{\ell_0}\|_\infty \lesssim (T_c - T)^{1/2}$.*

Proof. The gap equation, see (3.7), can be written as

$$\langle \alpha_{\ell_0}, (K_{T_c} + V) \alpha_{\ell_0} \rangle + \langle \alpha_{\ell_0}, B \alpha_{\ell_0} \rangle = \langle \alpha_{\ell_0}, A \alpha_{\ell_0} \rangle,$$

where we use the notation introduced in the proof of Lemma 4.5 but drop the subscript, i.e. $A = A_T$ and $B = B_T$ for brevity. Lemma 4.5 and the definition of T_c imply that

$$\langle \alpha_{\ell_0}, B \alpha_{\ell_0} \rangle \leq \langle \alpha_{\ell_0}, A \alpha_{\ell_0} \rangle \lesssim (T_c - T) \|\alpha_{\ell_0}\|_2^2. \quad (4.10)$$

From the combination of Lemma 4.6 and (4.10) we deduce that

$$\| (1+p^2)^{1/4} \hat{\alpha}_{\ell_0} \|_4^4 \lesssim (T_c - T) \|\alpha_{\ell_0}\|_2^2.$$

On the other hand, the $L^r(\mathbb{R}^2)$ -norm of $\hat{\alpha}$ is bounded from above by

$$\|\hat{\alpha}_{\ell_0}\|_r \leq \| (1+p^2)^{-1/4} \|_s \| (1+p^2)^{1/4} \hat{\alpha}_{\ell_0} \|_4,$$

where $r > 2$, due to the fact that we have to choose $s > 4$. Thus,

$$\|\hat{\alpha}_{\ell_0}\|_r^4 \lesssim (T_c - T) \|\hat{\alpha}_{\ell_0}\|_2^2. \quad (4.11)$$

Furthermore, we conclude from the relation between Δ_{ℓ_0} and α_{ℓ_0} given by the combination of (3.1) and (3.2) that

$$\|\Delta_{\ell_0}\|_\infty \lesssim \|\hat{V}\|_t \|\hat{\alpha}_{\ell_0}\|_r, \quad (4.12)$$

where we choose $r > 2$ and $t \in [1, 2)$ appropriately. Note that the gap equation in the form (3.6) implies that $\|\hat{\alpha}_{\ell_0}\|_2 \lesssim \|\Delta_{\ell_0}\|_\infty$. Together with (4.11) and (4.12) this finally shows that

$$\|\hat{\alpha}_{\ell_0}\|_2 \lesssim (T_c - T)^{1/4} \|\hat{\alpha}_{\ell_0}\|_2^{1/2}$$

and hence proves the first part of the claim. In order to get the estimate on $\|\Delta_{\ell_0}\|_\infty$, we go back to (4.11) and insert $\|\alpha_{\ell_0}\|_2 \lesssim (T_c - T)^{1/2}$. Together with (4.12) this yields the statement. \square

Let $T \in (0, T_c)$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Taken together, Lemma 4.5 and Lemma 4.7 show that

$$\begin{aligned} & \left\| (z - (K_{T_c} + V))^{-1} - \left(z - \left(K_T^{\Delta_{\ell_0}} + V \right) \right)^{-1} \right\| \\ & \leq \left\| (z - (K_{T_c} + V))^{-1} \right\| \left\| K_T^{\Delta_{\ell_0}} - K_{T_c} \right\| \left\| \left(z - \left(K_T^{\Delta_{\ell_0}} + V \right) \right)^{-1} \right\| \\ & \lesssim |\operatorname{Im}(z)|^{-2} (T_c - T)^{1/2}. \end{aligned}$$

In other words, $K_T^{\Delta_{\ell_0}} + V \rightarrow K_{T_c} + V$ for $T \rightarrow T_c$ in norm resolvent sense for an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$ and consequently for all $z \in \rho(K_{T_c} + V)$.

We are now prepared for the proof of Proposition 4.3.

Proof of Proposition 4.3. We consider the case $\ell_0 \neq 0$. The proof for the case $\ell_0 = 0$ is analogous. As illustrated in Figure 1, we have by assumption that $T_c = T_c(\ell_0)$ and that the lowest eigenvalue of $K_{T_c} + V$ is exactly twice degenerate. Note that in the case that $\ell_0 = 0$ the smallest eigenvalue is non-degenerate. From the convergence of $K_T^{\Delta_{\ell_0}} + V$ to $K_{T_c} + V$ in norm resolvent sense one concludes that the lowest eigenvalue of $K_T^{\Delta_{\ell_0}} + V$ is stable.

In particular, this tells us that there exists $\tilde{T} < T_c$ such that $K_T^{\Delta_{\ell_0}} + V$ with $T \in (\tilde{T}, T_c]$ has exactly two eigenvalues $\lambda_1(T), \lambda_2(T) \in \{z \in \mathbb{C} \mid |z| < r\}$ for some radius $r > 0$. Combining this with the fact that the Euler-Lagrange equation (3.8) of $\mathcal{F}_{\ell_0}^{\text{ti}}$ reads

$$(K_T^{\Delta_{\ell_0}} + V)\alpha = 0, \tag{4.13}$$

we conclude that $\lambda_1(T) = \lambda_2(T) = 0$. Having in mind that $K_T^{\Delta_{\ell_0}}$ is an increasing function of T and of Δ_{ℓ_0} , what we have seen by this argument is that the effects of these monotonicity properties exactly correspond. In other words, we have shown that there exists $\tilde{T} < T_c$ such that $K_T^{\Delta_{\ell_0}} + V$ is nonnegative for all $T \in [\tilde{T}, T_c]$. It is not hard to see that \tilde{T} can be chosen as pointed out in Remark 2.1. \square

Proof of Theorem 1. We know from Lemma 3.1 that for ℓ_0 determined by $T_c(\ell_0) = \max_{\ell \in 2\mathbb{N}} T_c(\ell)$ the functional $\mathcal{F}_{\ell_0}^{\text{ti}}$ has a minimizer $(\gamma_{\ell_0}, \sigma_{\ell_0})$. Proposition 4.2 and Proposition 4.3 show that for Γ_{ℓ_0} given by $(\gamma_{\ell_0}, \alpha_{\ell_0})$, with α_{ℓ_0} as in (2.6),

$$\mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_{\ell_0}) \geq 0,$$

holds for all $\Gamma \in \mathcal{D}$. Moreover, if $\mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_{\ell_0}) = 0$, then $\gamma = \gamma_{\ell_0}$ and $\alpha \in \ker(K_T^{\Delta_{\ell_0}} + V_y)$ by Proposition 4.2. Consequently, α takes the form $\alpha = \psi_1 \alpha_{\ell_0} + \psi_2 \alpha_{-\ell_0}$, where $\alpha_{\pm \ell_0}(p) = e^{\pm i \ell \varphi} \sigma_{\ell_0}(p)$ and ψ_1 and ψ_2 denote complex constants. It remains to show that either $\psi_1 = 0$ and $|\psi_2| = 1$ or $|\psi_1| = 1$ and $\psi_2 = 0$. Observe that, in particular,

$(\gamma_{\ell_0}, \alpha) \in \mathcal{D}^{\text{ti}}$ and as we know that \mathcal{F}^{ti} has a minimizer, we conclude that $(\gamma_{\ell_0}, \alpha)$ satisfies the Euler-Lagrange equation of \mathcal{F}^{ti} , that is

$$\gamma_{\ell_0}(p) = \frac{1}{2} - \frac{p^2 - \mu}{2K_T^\Delta(p)},$$

where $\Delta = \pi^{-1}\hat{V} * \hat{\alpha}$, see [18]. Hence $|\Delta|$ is a radial function and consequently either $\psi_1 = 0$ or $\psi_2 = 0$. In other words, $(\gamma_{\ell_0}, \sigma_{\ell_0}) \in \mathcal{D}_{\ell_0}$. Thus, in order to find minimizers of \mathcal{F} , it is sufficient to find the minimizers of $\mathcal{F}_{\ell_0}^{\text{ti}}$. As we know that $\mathcal{F}_{\ell_0}^{\text{ti}}$ has minimizers, the only thing left to show is that $(\gamma_{\ell_0}, \sigma_{\ell_0})$ is, up to a phase, the only minimizer of $\mathcal{F}_{\ell_0}^{\text{ti}}$. The fact that other possible minimizers $(\gamma_{\ell_0}, \psi\sigma_{\ell_0})$, for some $\psi \in \mathbb{C}$, have to satisfy the gap equation (3.7) of $\mathcal{F}_{\ell_0}^{\text{ti}}$ reads

$$\left(K_T^{\psi\Delta_{\ell_0}} + V_{\ell_0}\right)(\psi\sigma_{\ell_0}) = 0.$$

Together with the monotonicity of $K_T^{\psi\Delta_{\ell_0}}$ in ψ this implies that $|\psi| = 1$. \square

The proof of Theorem 2 is analogous to the proof of Theorem 1 with one exception.

Proof of Theorem 2. In case $\ell_0 = 0$ all given arguments also apply in the three-dimensional case. The only exception is Lemma 4.7, where we need to modify the assumptions slightly. One easily sees that $\hat{V} \in L^r(\mathbb{R}^3)$ with $r \in [1, 12/7)$ is a sufficient assumption in this case. \square

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